# The Meeting Problem

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## Introduction

This paper is the result of inquiry into the article “Mind the Gap” by Bannon and Bradley, from *The College Mathematics Journal.* Upon reading the paper, questions arose concerning the result of solution found when using integration to solve for the probability that some number of people meet when waiting for a given time over a certain interval. The article mentions the complexity of the geometric argument at higher dimensions. This paper explores the geometric solution at higher dimensions, as well as revisiting the solution via integration and joint probability density.

The first inspiration for a geometric argument came from the “Mind the Gap” [1] article. The authors state that the geometric solution is difficult to envision in higher dimension and that most would not be able to accomplish this feat. Further inspiration for the validity of a geometric argument came from the logic of “The Broken Spaghetti Noodle” by D’Andrea and Gomez [2]. The idea to solve with general waiting time for each person waiting on the park bench was inspired by “Meeting Probabilities” by K. S. Brown, the author of MathPages.com [3]. This paper provided a solution for the meeting of 3 people, but only gave the proposed solution for the general case of n people and did not provide evidence for this claim.

Let us look at the meeting problem, as given by Bannon and Bradley in the article “Mind the Gap”, from *The College Mathematics Journal* [1]*:*

> Ann and Ben plan to arrive at a park bench sometime between noon and 1:00 P.M. Each will wait up to 15 minutes for the other person to arrive. In the event that the other person does not arrive during the 15-minute period, then no meeting takes place. What is the probability that they meet?

We will give a geometric solution to this problem. Let x represent the time at which Ann arrives and let y represent the time at which Ben arrives. Also, let the interval from noon to 1:00 P.M. be represented by the unit interval, from [0,1]. Thus, the 15 minute wait can be represented by $\frac{1}{4}$. Now, we can represent the problem by the inequality

$|x - y| \leq \frac{1}{4}$. This implies $-\frac{1}{4} \leq x - y \leq \frac{1}{4}$. This yield the inequalities $y \leq x + \frac{1}{4}$ and $y \geq x - \frac{1}{4}$. Alternatively, we can make the assumption that x arrives before y and find the probability that they meet. Then, this probability will be the same as if we had assumed that y arrived before x. Thus, we can multiply the probability they meet when x arrives before y by 2 to find the probability that they meet in general. Now, if x arrives before y we know that $x \leq y$ and $y - x \leq \frac{1}{4}$ in order for the pair to meet. We can solve this graphically by plotting the region of the 2D plane that meets the conditions of this set of inequalities and finding the area of this region. The solutions to this set of inequalities are bounded by (0,0); (0,1); (1,0); (1,1) since x and y must meet within the interval [0,1]. From this, we obtain a unit square in the 2D plane. Then, the intersection points of our system of inequalities and the unit square are (0,0); $(0, \frac{1}{4})$; $(\frac{3}{4},1)$; (1,1). We then find the area of the solution region by splitting the region into a parallelogram and triangle. Notice that the segment from $(0,\frac{1}{4})$ to $(\frac{1}{4},\frac{1}{4})$ divides the region into these two shapes. Figure 1 below represents the graph of these inequalities within a unit square.

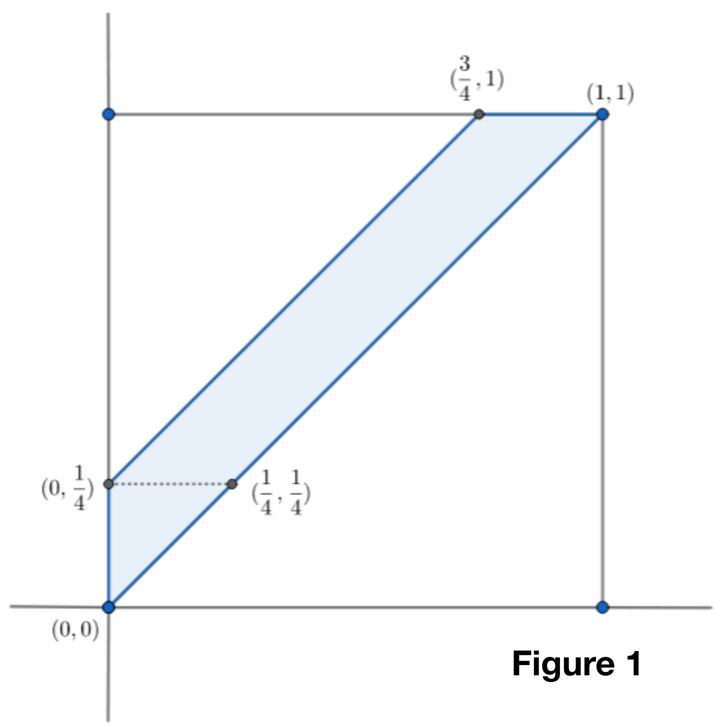


**Figure 1**

The area of the parallelogram is $\frac{1}{4} \cdot \frac{3}{4} = \frac{3}{16}$. The area of the triangle is $\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{32}$. Thus the total area of the shaded region in figure 1 is $\frac{3}{16} + \frac{1}{32} = \frac{7}{32}$. Since we assumed that x ≤ y, we then multiply this probability by 2 since the case where y ≤ x will also yield an area of $\frac{7}{32}$. Therefore, the probability of Ann and Ben meeting is $\frac{7}{16}$, since the area of the shaded region represents all coordinate points that are arrival times in which Ann and Ben would meet.

Now let us apply this same logic to the generalized meeting problem from the same article [1], which is:

> Suppose *n* people arrive at a given location randomly between noon and 1:00 P.M. If each one will wait up to 15 minutes for the others to arrive, what is the probability that a meeting occurs involving all of these *n* people?

We will first show how this can be done in the case of n = 3. Let the 3 times of arrival be called x, y, z. Assume that x ≤ y ≤ z. Observe that there are 3!=6 possible permutations for order of arrival between x, y, z, and thus we will need to multiply the resultant volume of our 3-dimensional solution region by 6 in order to obtain an accurate probability. We may also assume that z - x ≤ $\frac{1}{4}$, due to the order of the arrivals. We will use these inequalities to develop a solution region within a unit cube that represents the probability that x, y, z meet. If x = 0, then z ≤ $\frac{1}{4}$. This implies that 0 ≤ y ≤ $\frac{1}{4}$. We then obtain the following coordinate points as bounds of our shaded region: (0,0 0); (0,0,$\frac{1}{4}$); (0,$\frac{1}{4}$,$\frac{1}{4}$). Similarly, if z = 1, then x ≥ $\frac{3}{4}$. This implies that $\frac{3}{4}$ ≤ y ≤ 1. Thus we further bound the shaded region by the points: ($\frac{3}{4}$, $\frac{3}{4}$,1); ($\frac{3}{4}$,1,1); (1,1,1). The lines connecting (0,0,0) to (1,1,1); (0,0,$\frac{1}{4}$) to ($\frac{3}{4}$,1,1); and (0, $\frac{1}{4}$, $\frac{1}{4}$) to ($\frac{3}{4}$,$\frac{3}{4}$,1) then completes our shaded figure within the unit cube, since this is where our solution region intersects our top face of the unit cube. Thus, we can find the volume by cutting the figure into a triangular prism and a tetrahedron, which can be seen in figure 2 below.

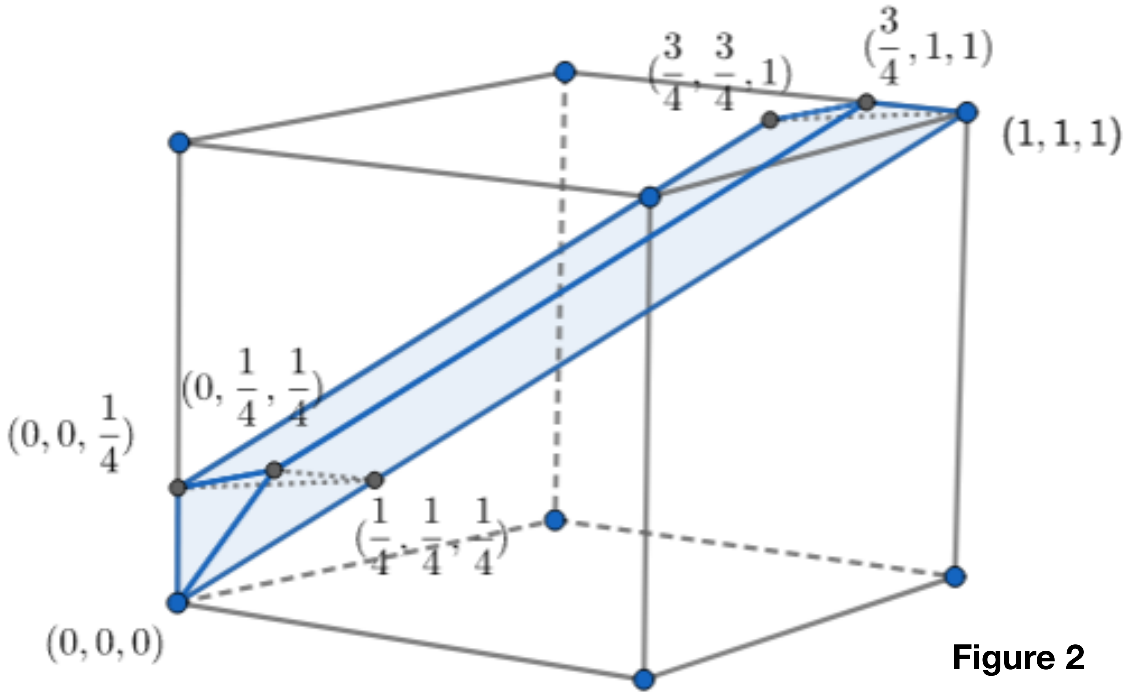

**Figure 2**

The triangular base of the prism has an area of $\frac{1}{32}$ , since the base and height of the triangle are both $\frac{1}{4}$. The height of the prism is $\frac{3}{4}$, since all points on the bottom base are at z = $\frac{1}{4}$ and all points on the top base are at z = 1. Thus, the volume of the prism is $\frac{1}{32} \cdot \frac{3}{4} = \frac{3}{128}$. The tetrahedron shares the triangular base with area = $\frac{1}{32}$ and all points at z = $\frac{1}{4}$, so the volume of the tetrahedron is $\frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{32} = \frac{1}{384}$. Hence, the entire volume of the shaded region in figure 2 is $\frac{3}{128} + \frac{1}{384} = \frac{5}{192}$. Since this was 1 of 6 permutations, we will have 6 3D shapes with the same volume as the shaded region in figure 2. Therefore, the probability for the generalized meeting problem when n = 3 is $6 \cdot \frac{5}{192} = \frac{5}{32}$.

Now, consider the case where n = 4. Let the 4 times of arrival be called x, y, z, w. Assume that x arrives first and w arrives last. This implies that $0 \leq x \leq y \leq z \leq w \leq 1$. Observe that there are 24 possible permutations for order of arrival between x, y, z, w and thus we will need to multiply the resultant volume of our 4-dimensional shaded region by 24 in order to obtain an accurate probability. We may also assume that $w - x \leq \frac{1}{4}$, due to the order of the arrivals. We will use these inequalities to develop a shaded region within a unit "cube" in 4 dimensions that represents the probability that x, y, z, w meet. The "cube" has vertices (0,0,0,0); (0,0,0,1); (0,0,1,0); (0,1,0,0); (1,0,0,0); (0,0,1,1); (0,1,0,1); (1,0,0,1); (0,1,1,0); (1,0,1,0); (1,1,0,0); (0,1,1,1); (1,0,1,1); (1,1,0,1); (1,1,1,0); (1,1,1,1); If x arrives at time 0, then w is at most $\frac{1}{4}$. This implies that $0 \leq y \leq \frac{1}{4}$ and $0 \leq z \leq \frac{1}{4}$. We then obtain the following coordinate points as bounds of our shaded region:

$(0,0,0,0)$; $(0,0,0,\frac{1}{4})$; $(0,0,\frac{1}{4},\frac{1}{4})$; $(0,\frac{1}{4},\frac{1}{4},\frac{1}{4})$. Similarly, if z = 1, then $x \geq \frac{3}{4}$. This implies that $\frac{3}{4} \leq y \leq 1$ and $\frac{3}{4} \leq z \leq 1$. Thus we further bound the shaded region by the points: $(\frac{3}{4},\frac{3}{4},\frac{3}{4},1)$; $(\frac{3}{4},\frac{3}{4},1,1)$; $(\frac{3}{4},1, 1, 1)$; (1,1,1,1). The lines connecting (0,0,0,0) to (1,1,1,1); $(0,0,0,\frac{1}{4})$ to $(\frac{3}{4},1,1,1)$; $(0,0,\frac{1}{4},\frac{1}{4})$ to $(\frac{3}{4},\frac{3}{4},1,1)$ and $(0, \frac{1}{4},\frac{1}{4},\frac{1}{4})$ to $(\frac{3}{4},\frac{3}{4},\frac{3}{4},1)$ then completes our shaded figure within the unit "cube". Thus, we can again find the volume by cutting the figure into a 4 dimensional "triangular prism" and a 4 dimensional "tetrahedron" as we did in the example for n=3. The "triangular prism" and "tetrahedron" again share a base, which has the vertices $(0,0,0,\frac{1}{4})$; $(0,0,\frac{1}{4},\frac{1}{4})$; $(0,\frac{1}{4},\frac{1}{4},\frac{1}{4})$; $(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$. Then the "volume" of the 4 dimensional "tetrahedron" is $\frac{1}{4}$ • Base • Height, where the base is the volume of the 3 dimensional tetrahedron from the n=3 example above, which is $\frac{1}{4} \cdot \frac{1}{32} \cdot \frac{1}{3} = \frac{1}{4^3 \cdot 6}$. Then since the height of the 4 dimensional "tetrahedron" is $\frac{1}{4}$, the volume is: $\frac{1}{4^5 \cdot 6}$. Notice that the base of the 4 dimensional "triangular prism" is at a height of

$w = \frac{1}{4}$. Thus the height of the 4 dimensional “triangular prism” is $\frac{3}{4}$, which implies that the “volume” of the “triangular prism” is $\frac{3}{4} \cdot \frac{1}{4^3 \cdot 6} = \frac{3}{4^4 \cdot 6}$. Therefore, the total “volume” of the 4 dimensional shaded region is $\frac{1}{4^5 \cdot 6} + \frac{3}{4^4 \cdot 6} = \frac{13}{4^5 \cdot 6}$. Then we can find the probability that x,y,z,w meet by multiplying this 4 dimensional “volume” by 24 since there are 24 permutations of the order of arrival for x,y,z,w. Thus the probability is: $\frac{13}{4^5 \cdot 6} \cdot 24 = \frac{13}{4^4} = \frac{13}{256}$.

We can extend this to the case where n people meet. Let the times of arrival be called $x_1$, $x_2, \ldots, x_n$. Assume that $x_1$ arrives first and $x_n$ arrives last. This implies that $0 \leq x_1 \leq x_2 \leq \ldots \leq x_n \leq 1$. Thus, there are n! permutations of the order of arrival for n arrival times and we will need to multiply our resulting probability using our initial assumption by n! in order to find the probability for this case in general. To further generalize our problem, we can solve the probability for a waiting time of q for each of the n people. That is, n people wait for time q on the interval [0,1].

We will apply the same technique in solving the probability for this general case. We will find the probability in dimension n using a “tetrahedron” and “triangular prism” that are bound within a unit “cube”. The “cube” has vertices which are all permutation using 0,1 for n entries in each coordinate.

For a waiting time q, we also have that $x_n - x_1 \leq q$ from our assumption $0 \leq x_1 \leq x_2 \leq \ldots \leq x_n \leq 1$. We then use this to obtain the boundary points of our solution region on the $x_1$ dimension face and the $x_n$ dimension face of our unit “cube”. When $x_1 = 0$, we obtain the points (0,0,…0); (0,0,…q); (0,0,…q,q); … (q,q,…q) as solution points to our inequalities. When $x_n = 1$, we obtain the points
(1-q, 1-q,…,1); (1-q, 1-q,…,1,1); … (1-q, 1,…,1); (1,1,…1).
Our region of solutions is then found by connecting each point on the face of the “cube” in the $x_1$ dimension with the corresponding point on the face of the “cube” in the $x_n$ dimension. We can now find the “volume” of the solution region by splitting that region into a “tetrahedron” and “triangular prism” of *n* dimension. The vertices of the common base of these 2 figures will then be given by all coordinate points where

$x_n$ = q, and all other entries in the coordinate point are either 0 or q. That is, (0,0,…q); (0,0,…q,q); …(0,q,…,q); (q,q,…,q) are the vertices of the common base. Then the volume of the "tetrahedron" can be found by calculating the determinant of the n *x* n matrix whose entries are the coordinates of the vertices of the figure. That is:

$$\text{Volume}_{\text{tetrahedron}} = \text{Det.} \begin{bmatrix} 0 & 0 & \dots & 0 & q \\ 0 & 0 & \dots & q & q \\ & & \vdots & & \\ q & q & \dots & q & q \end{bmatrix} = \frac{q^n}{n!}.$$

The "volume" of the "triangular prism" can be found by calculating the "volume" of the common base between the figure and multiplying it by the height of the "triangular prism". We can find the "volume" of the base by calculating the determinant of a n-1 *x* n-1 matrix whose entries are the coordinates of the base. That is:

$$\text{Volume}_{\text{base}} = \text{Det.} \begin{bmatrix} 0 & 0 & \dots & 0 & q \\ 0 & 0 & \dots & q & q \\ & & \vdots & & \\ q & q & \dots & q & q \end{bmatrix} = \frac{q^{n-1}}{(n-1)!}$$

$$\text{Volume}_{\text{Triangular Prism}} = (1-q) \cdot \frac{q^{n-1}}{(n-1)!}.$$

Now, we can find the probability by adding the "volume" of our two figures together, and multiplying the sum by the *n!* permutations of order of arrival for *n* people.

$$n! \left[ \frac{q^n}{n!} + (1-q) \cdot \frac{q^{n-1}}{(n-1)!} \right] = q^n + n(1-q)q^{n-1} = q^n + nq^{n-1} - nq^n = nq^{n-1} - (n-1)q^n$$

Therefore, the probability that *n* people meet between noon and 1:00 p.m., where each person waits for *q* time is $nq^{n-1} - (n-1)q^n$.

The same solution can be found through integration when using a joint probability density function for the first person meeting and last people to arrive on the bench. Again, let the times of arrival be called $x_1, x_2, \ldots, x_n$. Also, let the amount of time each person waits be q, $X = min(x_1, x_2, ..., x_n)$ and $Y = max(x_1, x_2, ..., x_n)$. Then we need $Y - X \leq q$ in order for the *n* people to meet. Visually, we can represent this idea with figure 3 below:

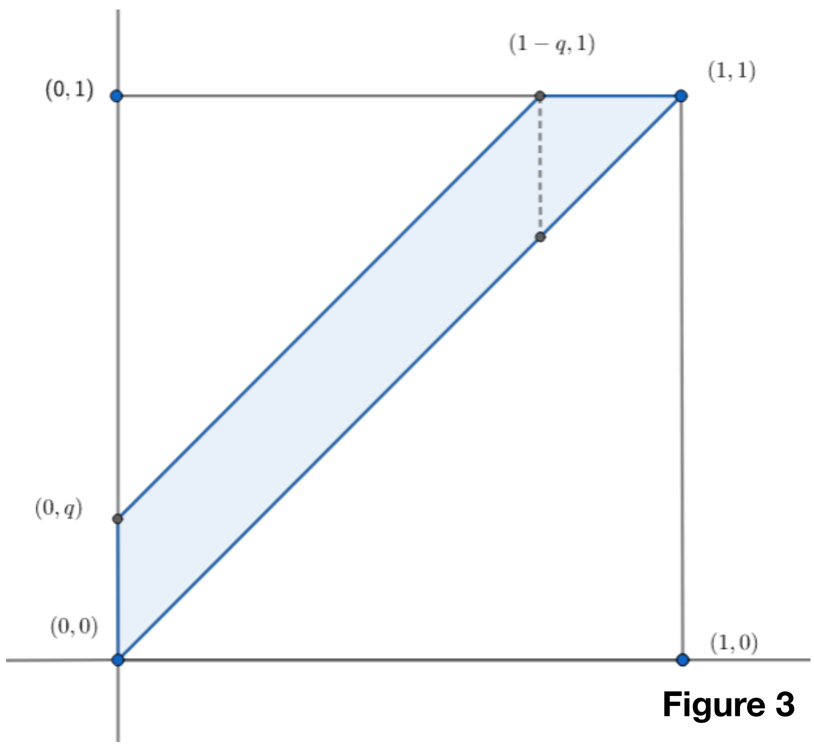


**Figure 3**

Thus, we can represent the probability with a joint probability density function :

$P(X \leq x, Y \leq y) = P(Y \leq y) - P(X > x, Y \leq y)$. This implies that:

$P(x_1 \leq y, x_2 \leq y, ..., x_n \leq y) - P(x < x_1 \leq y, x < x_2 \leq y, ..., x < x_n \leq y)$

Since each $x_i$ is independent, we this implies:

$$\prod_{i=1}^{n} P(x_i \leq y) - \prod_{i=1}^{n} P(x < x_i \leq y)$$

Notice that $P(x_i \leq y) = y$ and $P(x < x_i \leq y) = y - x$. Hence, our joint probability density function is $y^n - (y - x)^n$. Then by taking the derivative of our function with respect to x and to y, we find our joint density:

$$\frac{d}{dx}\frac{d}{dy}(y^n - (y - x)^n) = n(n-1)(y - x)^{n-2}$$

Now, using our joint density, we can find the area of the shaded polygon from figure 3 by integrating:

$$\int_0^{1-q}\int_x^{x+q} n(n-1)(y-x)^{n-2}dydx + \int_{1-q}^{1}\int_x^{1} n(n-1)(y-x)^{n-2}dydx$$

This yields, $nq^{n-1} - nq^n + q^n = nq^{n-1} - (n-1)q^n$, which is the same solution as the geometric argument presented above.

While the "Mind the Gap" article was a very interesting paper, we unfortunately can conclude that in contains at least one error in the argument of the general case of the meeting problem. This paper shows, using two separate methods, the probability that *n* people meet when waiting for a given time over a unit interval. Both methods yield the same result and provide a definite solution, as opposed to approximations.

References

[1] V. Bannon and R. Bradley, Mind The Gap, The College Mathematics Journal, 39(2), 95-101 (2008).

[2] C. D’Andrea and E. G´omez, The Broken Spagetti Noodle, Amer. Math. Monthly, 133, 555-557 (2006).

[3]Brown, K. S. (n.d.). Meeting Probabilities. Retrieved March 12, 2020, from https://www.mathpages.com/home/kmath124/kmath124.htm